

# CONTROL OF FUSION BY ABELIAN SUBGROUPS OF THE HYPERFOCAL SUBGROUP

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**ABSTRACT.** We prove that an isomorphism between saturated fusion systems over the same finite  $p$ -group is detected on the elementary abelian subgroups of the hyperfocal subgroup if  $p$  is odd, and on the abelian subgroups of the hyperfocal subgroup of exponent at most 4 if  $p = 2$ . For odd  $p$ , this has implications for mod  $p$  group cohomology.

## 1. INTRODUCTION

In 1971, Quillen [18] published two articles relating properties of the mod  $p$  cohomology ring of a group  $G$  to the elementary abelian  $p$ -subgroups of  $G$ . The results hold for any prime  $p$  and any group  $G$  which is a compact Lie group (e.g. a finite group). Quillen studied in particular varieties of mod  $p$  cohomology rings and proved a stratification theorem stating that the variety of the mod  $p$  cohomology ring of  $G$  can be broken up into pieces corresponding to the  $G$ -conjugacy classes of elementary abelian  $p$ -subgroups of  $G$ .<sup>1</sup> Therefore, it is of interest to study conjugacy relations between elementary abelian  $p$ -subgroups. From now on we assume that  $G$  is finite and  $H$  is a subgroup of  $G$  of index prime to  $p$ . For any two subgroups  $A$  and  $B$  of  $G$ , we write  $\text{Hom}_G(A, B)$  for the set of group homomorphisms from  $A$  to  $B$  that are obtained via conjugation by an element of  $G$ . As a consequence of Quillen's stratification theorem,  $H$  controls fusion of elementary abelian subgroups in  $G$ , if the inclusion map from  $H$  to  $G$  induces an isomorphism between the varieties of the mod  $p$  cohomology rings of  $H$  and  $G$ . Here we say that the subgroup  $H$  controls fusion of elementary abelian subgroups in  $G$  if  $\text{Hom}_H(A, B) = \text{Hom}_G(A, B)$  for all elementary abelian subgroups  $A$  and  $B$  of  $H$ . Similarly we say that  $H$  controls  $p$ -fusion in  $G$  if  $\text{Hom}_H(A, B) = \text{Hom}_G(A, B)$  for all  $p$ -subgroups  $A$  and  $B$  of  $H$ . By the Cartan–Eilenberg stable elements formula [9, XII.10.1], the inclusion map from  $H$  to  $G$  induces an isomorphism in mod  $p$  group cohomology if  $H$  controls fusion in  $G$ . Together with Quillen's fundamental results, this motivates the study of connections between control of fusion of elementary abelian subgroups and control of  $p$ -fusion.

If  $H = S$  is a Sylow  $p$ -subgroup of  $G$  and  $p$  is odd, Quillen [17] proved as a first illustration of his theory that  $G$  is nilpotent if the inclusion map from  $S$  to  $G$  induces an isomorphism between the corresponding varieties. We recall that, by a classical theorem of Frobenius,  $G$  is nilpotent if and only if  $S$  controls fusion in  $G$ . So Quillen showed that  $S$  controls fusion in  $G$  if  $S$  controls fusion of elementary abelian subgroups. Variations of this theorem were proved in [12, 7, 10, 13, 8, 2], but all maintaining the hypothesis that  $H = S$  is a Sylow  $p$ -subgroup. Only relatively recently, Benson, Grodal and the first author of this paper

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<sup>1</sup>More precisely, Quillen studied the variety of the commutative subring of  $H^*(G; \mathbb{F}_p)$  of elements of even degree. However, his stratification theorem holds similarly for the variety of  $H^*(G; \mathbb{F}_p)$ ; see Remark 3.2.

proved a result that holds more generally for any subgroup  $H$  of index prime to  $p$ ; see [5]. More precisely, it is shown that  $H$  controls fusion in  $G$  (and thus the inclusion map from  $H$  to  $G$  induces an isomorphism in mod  $p$  group cohomology), if the inclusion map induces an isomorphism between the corresponding varieties, i.e. if  $H$  controls fusion of elementary abelian subgroups of  $G$ . This is obtained as a consequence of a theorem that is stated and proved for saturated fusion systems; see [5, Theorem B]. In this short note, we point out that actually a slightly stronger version of this theorem holds. We refer the reader to [1, Part I] for an introduction to fusion systems.

**Theorem A** (Small exponent abelian subgroups of the hyperfocal subgroup control fusion). *Let  $\mathcal{G} \subseteq \mathcal{F}$  be two saturated fusion systems over the same finite  $p$ -group  $S$ . Suppose that  $\text{Hom}_{\mathcal{G}}(A, B) = \text{Hom}_{\mathcal{F}}(A, B)$  for all  $A, B \leq \text{hfp}(\mathcal{F})$  with  $A, B$  elementary abelian if  $p$  is odd, and abelian of exponent at most 4 if  $p = 2$ . Then  $\mathcal{G} = \mathcal{F}$ .*

If one replaces  $\text{hfp}(\mathcal{F})$  by  $S$ , then the above theorem coincides with [5, Theorem B]. We recall that the hyperfocal subgroup  $\text{hfp}(\mathcal{F})$  is the subgroup of  $S$  generated by all elements of the form  $x^{-1}\varphi(x)$  where  $x \in Q$  and  $\varphi \in O^p(\text{Aut}_{\mathcal{F}}(Q))$  for some subgroup  $Q$  of  $S$ . If  $\mathcal{F} = \mathcal{F}_S(G)$  is the fusion system of a finite group  $G$  with Sylow  $p$ -subgroup  $S$ , then Puig's hyperfocal subgroup theorem [16, §1.1] states that  $\text{hfp}(\mathcal{F}) = O^p(G) \cap S$ . In the situation of Theorem A, Quillen's example  $Q_8 \leq Q_8 : C_3$  shows that it is indeed not enough to consider only elementary abelian subgroups for  $p = 2$ .

A fusion system  $\mathcal{F}$  on  $S$  is called nilpotent if  $\mathcal{F} = \mathcal{F}_S(S)$ . Restricting attention to subgroups of the hyperfocal subgroup is motivated by a theorem of the second author of this paper together with Zhang, which characterizes  $p$ -nilpotency of a saturated fusion system  $\mathcal{F}$  by the fusion on certain subgroups of the hyperfocal subgroup of  $\mathcal{F}$ ; see [14]. Another motivation comes from work of Ballester-Bolinches, Ezquerro, Su and Wang [2] showing that, in certain special cases, fusion is detected on the subgroups of the focal subgroup of  $\mathcal{F}$  which are cyclic of order  $p$  or 4. We show here that in Theorem A and C of [2], the focal subgroup can actually be replaced by the hyperfocal subgroup. More precisely, we prove the following theorem which gives in particular a new characterization of nilpotent fusion systems:

**Theorem B.** *Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ , and let  $\mathcal{G} = N_{\mathcal{F}}(S)$  or  $\mathcal{G} = \mathcal{F}_S(S)$ . Suppose that  $\text{Hom}_{\mathcal{G}}(A, B) = \text{Hom}_{\mathcal{F}}(A, B)$  for all  $A, B \leq \text{hfp}(\mathcal{F})$  which are cyclic subgroups of order  $p$  or 4. Then  $\mathcal{G} = \mathcal{F}$ .*

We remark that, in general, it is not the case that the subgroup  $H$  controls fusion in  $G$  if it controls fusion on cyclic subgroups of order  $p$  for odd  $p$ , or on subgroups of order at most 4 for  $p = 2$ . This is not even the case if  $G$  has a normal Sylow  $p$ -subgroup as the following example shows: Let  $n$  be an integer such that  $n \geq 2$  and  $p$  does not divide  $n$ . Let  $S$  be the field of order  $p^n$ , so that  $S$  under addition forms in particular an elementary abelian group of order  $p^n$ . Note that every non-zero element of  $S$  induces a group automorphism of  $S$  via multiplication. Let  $D$  be the group of all these automorphisms. Then  $D$  is a subgroup of  $\text{Aut}(S)$  of order  $p^n - 1$  acting freely and transitively on the non-trivial elements of  $S$ . Let  $\sigma$  be the Frobenius automorphism of the field  $S$ . Then  $\sigma$  has order  $n$  and is also a group automorphism of  $S$ . Moreover,  $\sigma$  normalizes  $D$ , as conjugation by  $\sigma$  takes every element of  $D$  to its  $p$ th power. Hence,  $\hat{D} = D \rtimes \langle \sigma \rangle$  is a group of order  $(p^n - 1)n$ . Since  $p$  does not divide  $n$ , it follows that  $S$  is a normal Sylow  $p$ -subgroup of  $G := S \rtimes \hat{D}$ . Moreover,  $H := S \rtimes D$

is a subgroup of  $G$  of index prime to  $p$ . Note also that  $S = [S, D] = \text{hyp}(\mathcal{F}_S(G))$ . Let  $\mathcal{V}$  be the set of subgroups of  $S$  of order  $p$ . Then  $\mathcal{V}$  has  $\frac{p^n-1}{p-1}$  elements. As  $D$  acts freely and transitively on the non-trivial elements of  $S$ , it follows that  $D$  acts also transitively on  $\mathcal{V}$ , and that  $C_D(A) = 1$  for all  $A \in \mathcal{V}$ . Thus  $|\text{Aut}_D(A)| = |N_D(A)| = \frac{|D|}{|\mathcal{V}|} = p-1$  for every  $A \in \mathcal{V}$ . As any two elements of  $\mathcal{V}$  are conjugate under  $D$ , it follows  $|\text{Hom}_D(A, B)| = p-1$  for all  $A, B \in \mathcal{V}$ . Thus,  $\text{Hom}_H(A, B) = \text{Hom}_D(A, B)$  is the set  $\text{Inj}(A, B)$  of injective group homomorphism from  $A$  to  $B$ . As  $\text{Hom}_H(A, B) \subseteq \text{Hom}_G(A, B) \subseteq \text{Inj}(A, B)$ , this implies  $\text{Hom}_H(A, B) = \text{Hom}_G(A, B)$  for all  $A, B \in \mathcal{V}$ . So  $H$  controls fusion in  $G$  of the cyclic subgroups of order  $p$  (and thus for  $p = 2$  also of the cyclic subgroups of order at most 4). However, as  $D \neq \hat{D}$ , the subgroup  $H$  does not control fusion in  $G$ .

We conclude by stating a version of Theorem A in terms of varieties of cohomology rings. We continue to assume that  $G$  is a finite group and we fix moreover an algebraically closed field  $\Omega$  of prime characteristic  $p$ . We either set  $k = \Omega$  or  $k = \mathbb{F}_p$ . Moreover, set  $H^*(G) := H^*(G, k)$  and define the variety  $V_G$  to be the variety  $\text{Hom}_k(H^*(G), \Omega)$  of  $k$ -algebra homomorphisms from  $H^*(G)$  to  $\Omega$ ; see Remark 3.2 for alternative definitions of  $V_G$ . Then every  $k$ -algebra homomorphism  $\alpha : H^*(G) \rightarrow H^*(H)$  induces a map of varieties  $\alpha^* : V_H \rightarrow V_G$  by sending any homomorphism  $\beta \in V_H = \text{Hom}_k(H^*(H), \Omega)$  to  $\beta \circ \alpha \in V_G = \text{Hom}_k(H^*(G), \Omega)$ . For an arbitrary subgroup  $H$  of  $G$ , we write  $\text{res}_{G,H} : H^*(G) \rightarrow H^*(H)$  for the map induced by the inclusion map  $H \rightarrow G$ , and hence  $\text{res}_{G,H}^* : V_H \rightarrow V_G$  for the corresponding map of varieties.

If  $H$  is a subgroup of  $G$  containing a Sylow  $p$ -subgroup  $S$  of  $G$ , then we have the inclusion maps  $S \cap O^p(G) \hookrightarrow H \hookrightarrow G$  which induce the following maps of varieties:

$$V_{S \cap O^p(G)} \xrightarrow{\text{res}_{H, S \cap O^p(G)}^*} V_H \xrightarrow{\text{res}_{G,H}^*} V_G$$

So in particular, we can consider the restriction of the map  $\text{res}_{G,H}^* : V_H \rightarrow V_G$  to the subvariety  $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$  of  $V_H$ . If  $p$  is an odd prime and  $H$  is a subgroup of  $G$  of index prime to  $p$ , then the results in [5] say basically that  $H$  controls fusion in  $G$  if  $\text{res}_{G,H}^* : V_H \rightarrow V_G$  is an isomorphism of varieties. Theorem A implies a slightly stronger statement which is stated in the next theorem. Notice that a subgroup  $H$  of  $G$  has index prime to  $p$  if and only if  $H$  contains a Sylow  $p$ -subgroup of  $G$ .

**Theorem C.** *Let  $G$  be a finite group, let  $p$  be an odd prime, and let  $H$  be a subgroup of  $G$  containing a Sylow  $p$ -subgroup  $S$  of  $G$ . Suppose the restriction of the map  $\text{res}_{G,H}^*$  to  $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$  is injective. Then  $H$  controls fusion in  $G$  and the restriction map  $\text{res}_{G,H} : H^*(G) \rightarrow H^*(H)$  is an isomorphism.*

Note that Theorem C says in particular that the map  $\text{res}_{G,H}^* : V_H \rightarrow V_G$  is an isomorphism of varieties if its restriction to  $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$  is injective. One sees easily that the converse of Theorem C holds as well: If  $\text{res}_{G,H} : H^*(G) \rightarrow H^*(H)$  is an isomorphism then  $\text{res}_{G,H}^* : V_H \rightarrow V_G$  is an isomorphism. In particular, the restriction of  $\text{res}_{G,H}^*$  to  $\text{res}_{H, S \cap O^p(G)}^* V_{S \cap O^p(G)}$  is injective.

We remark also that a theorem analogous to Theorem C can be proved for saturated fusion systems rather than for groups. For more details, we refer the reader to Remark 3.5.

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## 2. PROOF OF THEOREM A AND THEOREM B

The proof of Theorem A is very similar to the proof of Theorem B in [5]. We need the following variation of [5, Theorem 2.1].

**Theorem 2.1.** *Let  $P$  be a finite  $p$ -group and let  $G$  be a subgroup of  $\text{Aut}(P)$  containing the group  $\text{Inn}(P)$  of inner automorphisms. Then there exists a  $G$ -invariant subgroup  $D$  of  $[P, O^p(G)]$ , of exponent  $p$  if  $p$  is odd and exponent at most 4 if  $p = 2$ , such that  $[D, P] \leq Z(D)$ , and such that every non-trivial  $p'$ -automorphism in  $G$  restricts to a non-trivial  $p'$ -automorphism of  $D$ . Furthermore, for any such  $D$  and any maximal (with respect to inclusion) abelian subgroup  $A$  of  $D$  it follows that  $A \trianglelefteq P$  and  $C_G(A)$  is a  $p$ -group.*

*Proof.* By [5, Theorem 2.1], there exists a characteristic subgroup  $D_1$  of  $P$ , of exponent  $p$  if  $p$  is odd and exponent at most 4 if  $p = 2$ , such that  $[D_1, P] \leq Z(D_1)$ , and such that every non-trivial  $p'$ -automorphism of  $P$  restricts to a non-trivial  $p'$ -automorphism of  $D_1$ . Set  $D := [D_1, O^p(G)]$ . As  $D_1$  is  $G$ -invariant and as  $O^p(G)$  is normal in  $G$ , the subgroup  $D$  is  $G$ -invariant. In particular, as  $\text{Inn}(P) \leq G$  by assumption, we have  $[D, P] \leq D$ . Using  $[D_1, P] \leq Z(D_1)$  we obtain thus  $[D, P] \leq [D_1, P] \cap D \leq Z(D_1) \cap D \leq Z(D)$ . If  $\varphi$  is a  $p'$ -automorphism of  $P$  with  $\varphi|_D = \text{Id}_D$  then  $[D, \varphi] = 1$  and  $[D_1, \varphi] \leq [D_1, O^p(G)] = D$ . Thus, by [11, Theorem 5.3.6], we have  $[D_1, \varphi] = [D_1, \varphi, \varphi] \leq [D, \varphi] = 1$  and  $\varphi|_{D_1} = \text{Id}_{D_1}$ . Because of the way  $D_1$  was chosen, this implies that  $\varphi = \text{Id}_P$ . So we have shown that every non-trivial  $p'$ -automorphism in  $G$  restricts to a non-trivial automorphism of  $D$ .

For the last part let  $A$  be a maximal subgroup of  $D$  with respect to inclusion. Then  $[A, P] \leq Z(D) \leq A$  and thus  $A \trianglelefteq P$ . Furthermore, if  $B \leq C_G(A)$  is a  $p'$ -subgroup, then  $A \times B$  acts on  $D$ . Since  $A$  is maximal abelian, it follows  $C_D(A) = A \leq C_D(B)$ . Thompson's  $A \times B$ -lemma [11, Theorem 5.3.4] now says that  $[D, B] = 1$  and so  $B = 1$ . Since  $B$  was arbitrary, it follows that  $C_G(A)$  is a  $p$ -group.  $\square$

We need the following crucial lemma, which is [5, Main Lemma 2.4].

**Lemma 2.2.** *Let  $\mathcal{G} \subseteq \mathcal{F}$  be two saturated fusion systems on the same finite  $p$ -group  $S$ , and  $P \leq S$  an  $\mathcal{F}$ -centric and fully  $\mathcal{F}$ -normalized subgroup, with  $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}_{\mathcal{G}}(R)$  for every  $P < R \leq N_S(P)$ . Suppose that there exists a subgroup  $Q \trianglelefteq P$  with  $\text{Hom}_{\mathcal{F}}(Q, S) = \text{Hom}_{\mathcal{G}}(Q, S)$ . Then  $\text{Aut}_{\mathcal{F}}(P) = \langle \text{Aut}_{\mathcal{G}}(P), C_{\text{Aut}_{\mathcal{F}}(P)}(Q) \rangle$ .*

*Proof of Theorem A.* By Alperin's fusion theorem [1, Theorem I.3.6],  $\mathcal{F}$  is generated by  $\mathcal{F}$ -automorphisms of fully  $\mathcal{F}$ -normalized  $\mathcal{F}$ -centric subgroups. We want to show that  $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{G}}(P)$  for all  $P \leq S$ . By induction on  $|S : P|$ , we can assume that  $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}_{\mathcal{G}}(R)$  for all  $R \leq S$  with  $|R| > |P|$ . Furthermore, by Alperin's fusion theorem, we can choose  $P$  to be fully  $\mathcal{F}$ -normalized and  $\mathcal{F}$ -centric. By Theorem 2.1, we can pick an  $\text{Aut}_{\mathcal{F}}(P)$ -invariant subgroup  $D$  of  $[P, O^p(\text{Aut}_{\mathcal{F}}(P))]$ , of exponent  $p$  if  $p$  is odd and of exponent at most 4 if  $p = 2$ , such that every non-trivial  $p'$ -automorphism  $\varphi \in \text{Aut}_{\mathcal{F}}(P)$  restricts to a non-trivial automorphism of  $D$  and, for any maximal (with respect to inclusion) abelian subgroup  $A$  of  $D$ ,  $A \trianglelefteq P$  and  $C_{\text{Aut}_{\mathcal{F}}(P)}(A)$  is a  $p$ -group. As  $P$  is fully  $\mathcal{F}$ -normalized,  $\text{Aut}_S(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ , and so if we replace  $A$  by a conjugate of  $A$  under  $\text{Aut}_{\mathcal{F}}(P)$ , we can arrange that  $C_{\text{Aut}_{\mathcal{F}}(P)}(A) \leq \text{Aut}_S(P) \leq \text{Aut}_{\mathcal{G}}(P)$ . As  $D$  has exponent  $p$  if  $p$  is odd and exponent at most 4 if  $p = 2$ , we have by assumption  $\text{Hom}_{\mathcal{F}}(A, S) = \text{Hom}_{\mathcal{G}}(A, S)$ . So by Lemma 2.2 applied with  $A$  in place of  $Q$ , we obtain that  $\text{Aut}_{\mathcal{F}}(P) = \langle \text{Aut}_{\mathcal{G}}(P), C_{\text{Aut}_{\mathcal{F}}(P)}(A) \rangle = \text{Aut}_{\mathcal{G}}(P)$  as wanted.  $\square$

Let  $\mathcal{P}$  be a set of representatives of the  $\mathcal{F}$ -conjugacy classes of  $\mathcal{F}$ -essential subgroups. A version of the Alperin–Goldschmidt Theorem for fusion systems states that  $\mathcal{F}$  is generated by the  $\mathcal{F}$ -automorphism groups of the elements of  $\mathcal{P} \cup \{S\}$ . Analyzing what is used in the proof above, one sees that we only need the following condition in Theorem A: For every  $P \in \mathcal{P} \cup \{S\}$  and every abelian subgroup  $A$  of the commutator subgroup  $[P, O^p(\text{Aut}_{\mathcal{F}}(P))]$  which is of exponent  $p$  or 4, we have  $\text{Hom}_{\mathcal{F}}(A, S) = \text{Hom}_G(A, S)$ .

The proof of Theorem B is essentially the same as the one of [2, Theorem A] except that we use Theorem 2.1 instead of [5, Theorem 2.1]. Essentially, Theorem B is a consequence of the following lemma:

**Lemma 2.3.** *Let  $\mathcal{F}$  be a saturated fusion systems over a finite  $p$ -group  $S$ . Suppose that  $\text{Hom}_{\mathcal{F}}(A, B) \subseteq \text{Hom}_{N_{\mathcal{F}}(S)}(A, B)$  for all subgroups  $A, B \leq \text{hnp}(\mathcal{F})$  which are cyclic of order  $p$  or 4. Then  $\mathcal{F} = N_{\mathcal{F}}(S)$ .*

*Proof.* Suppose that  $Q$  is an  $\mathcal{F}$ -essential subgroup. Then by definition,  $Q$  is in particular fully normalized and thus  $\text{Aut}_S(Q)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ . By Theorem 2.1, there is an  $\text{Aut}_{\mathcal{F}}(Q)$ -invariant subgroup  $D \leq [Q, O^p(\text{Aut}_{\mathcal{F}}(Q))] \leq Q \cap \text{hnp}(\mathcal{F})$  such that every non-trivial  $p'$ -element of  $\text{Aut}_{\mathcal{F}}(Q)$  restricts to a non-trivial automorphism of  $D$ , and  $D$  is of exponent  $p$  or 4. Let  $Z_i(S)$  be the  $i$ -th center of  $S$  and  $D_i = D \cap Z_i(S)$ . We argue now that  $D_i$  is  $\text{Aut}_{\mathcal{F}}(Q)$ -invariant: For every  $x \in D_i$  and any  $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ ,  $\varphi|_{\langle x \rangle}$  extends by hypothesis to an element of  $\text{Aut}_{\mathcal{F}}(S)$  which clearly normalizes  $Z_i(S)$ . As  $\varphi$  normalizes  $D$ , it follows  $\varphi(x) \in Z_i(S) \cap D = D_i$ . So  $D_i$  is indeed  $\text{Aut}_{\mathcal{F}}(Q)$ -invariant. Thus, for some  $n \in \mathbb{N}$ , the series  $1 = D_0 \leq D_1 \leq \dots \leq D_n = D$  is  $\text{Aut}_{\mathcal{F}}(Q)$ -invariant. So the stabilizer  $H$  of this series (i.e. the set of elements in  $\text{Aut}_{\mathcal{F}}(Q)$  acting trivially on  $D_i/D_{i-1}$  for each  $i \leq n$ ) forms a normal subgroup of  $\text{Aut}_{\mathcal{F}}(Q)$ . For any  $p'$ -element  $\varphi$  of  $H$ , we have  $\varphi|_D = \text{Id}_D$  by [11, Theorem 5.3.2], and thus  $\varphi = \text{Id}_Q$  by the choice of  $D$ . Therefore, the stabilizer  $H$  is a  $p$ -group and so  $H \leq O_p(\text{Aut}_{\mathcal{F}}(Q))$ . Since  $\text{Aut}_S(Q)$  stabilizes the series  $D_0 \leq D_1 \leq \dots \leq D_n = D$ , it follows that  $\text{Aut}_S(Q) = O_p(\text{Aut}_{\mathcal{F}}(Q))$ , which is a contradiction as every  $\mathcal{F}$ -essential subgroup is centric and radical. Hence there is no  $\mathcal{F}$ -essential subgroup. Thus,  $\mathcal{F} = N_{\mathcal{F}}(S)$  by Alperin’s fusion theorem [1, Theorem I.3.6].  $\square$

*Proof of Theorem B.* Lemma 2.3,  $\mathcal{F} = N_{\mathcal{F}}(S)$ . So for  $\mathcal{G} = N_{\mathcal{F}}(S)$  the assertion follows immediately. Assume now  $\mathcal{G} = \mathcal{F}_S(S)$ . As  $\mathcal{F} = N_{\mathcal{F}}(S)$ , it is sufficient to show that  $\text{Aut}_{\mathcal{F}}(S) = \text{Inn}(S)$ . By Theorem 2.1, there is an  $\text{Aut}_{\mathcal{F}}(S)$ -invariant subgroup  $D \leq [S, O^p(\text{Aut}_{\mathcal{F}}(S))] \leq S \cap \text{hnp}(\mathcal{F})$  such that every non-trivial  $p'$ -element of  $\text{Aut}_{\mathcal{F}}(S)$  restricts to a non-trivial automorphism of  $D$ , and  $D$  is of exponent  $p$  or 4. Let  $D_i = D \cap Z_i(S)$  and  $n \in \mathbb{N}$  such that  $D_n = D$ . By hypothesis, every element of  $\text{Aut}_{\mathcal{F}}(S)$  acts on every element of  $D$  as conjugation by an element of  $S$ . Hence,  $\text{Aut}_{\mathcal{F}}(S)$  stabilizes the series  $1 = D_0 \leq D_1 \leq \dots \leq D_n = D$  and is thus a  $p$ -group by [11, Theorem 5.3.2]. Since  $\text{Inn}(S) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(S))$ , it follows  $\text{Aut}_{\mathcal{F}}(S) = \text{Inn}(S)$  as required.  $\square$

### 3. PROOF OF THEOREM C

Throughout, assume that  $G$  is a finite group and that  $\Omega$  is an algebraically closed field of prime characteristic  $p$ . Let  $H^*(G)$  and  $V_G$  be as in the introduction. Recall that, for any subgroup  $H$  of  $G$ , we write  $\text{res}_{G,H}: H^*(G) \rightarrow H^*(H)$  for the map induced by the inclusion map from  $H$  to  $G$ , and  $\text{res}_{G,H}^*: V_H \rightarrow V_G$  for the corresponding map of varieties.

For the proof of Theorem C we will need some more notation: For every elementary abelian  $p$ -group  $A$ , we set

$$V_A^+ := V_A \setminus \bigcup_{A' < A} \text{res}_{A,A'}^* V_{A'}.$$

If  $A$  is an elementary abelian subgroup of  $G$ , set

$$V_{G,A}^+ = \text{res}_{G,A}^* V_A^+.$$

We start with the following elementary observation:

**Remark 3.1.** Let  $A \leq K \leq G$  such that  $A$  is elementary abelian. Then  $\text{res}_{G,K}^* V_{K,A}^+ = V_{G,A}^+$ .

*Proof.* As  $\text{res}_{G,K}^* \circ \text{res}_{K,A}^* = \text{res}_{G,A}^*$ , we have  $V_{G,A}^+ = \text{res}_{G,A}^* V_A^+ = \text{res}_{G,K}^*(\text{res}_{K,A}^* V_A^+) = \text{res}_{G,K}^* V_{K,A}^+$ .  $\square$

**Remark 3.2.** Write  $H^{ev}(G)$  for the subring of  $H^*(G)$  of elements of even degree. If  $k = \mathbb{F}_p$  notice that the  $k$ -algebra homomorphisms from  $H^*(G)$  to  $\Omega$  are the same as the ring homomorphisms from  $H^*(G)$  to  $\Omega$ . So if  $k = \mathbb{F}_p$  then, upon replacing  $H^*(G)$  by  $H^{ev}(G)$  if  $p$  is odd, the variety  $V_G$  corresponds to the variety  $H_G(X)(\Omega)$  studied by Quillen [18] in the special case that  $X$  is a point. If  $k = \Omega$ , it follows from Hilbert's Nullstellensatz that  $V_G$  is homeomorphic to the maximal ideal spectrum of  $H^*(G)$  via the map sending every homomorphism in  $V_G$  to its kernel; see Theorem 5.4.2 and the surrounding discussion in [3]. So again upon replacing  $H^*(G)$  by  $H^{ev}(G)$ , the variety  $V_G$  as defined in this paper corresponds to the variety  $V_G$  as defined by Benson [3].

It is common to study the variety of  $H^{ev}(G)$  rather than the variety of  $H^*(G)$ , because  $H^{ev}(G)$  is commutative, whereas  $H^*(G)$  is only graded commutative, and texts on commutative algebra are written for strictly commutative rings. As pointed out by Benson [4, p.9], the results from commutative algebra which are needed in the theory hold accordingly for graded commutative rings. Moreover, it is pointed out that any graded commutative ring  $A$  is commutative modulo its nilradical, and every element of odd degree lies in the nilradical if  $p$  is odd. So writing  $\mathfrak{Nil}$  for the nilradical of  $H^*(G)$ , it follows that  $H^*(G)/\mathfrak{Nil}$  is isomorphic to  $H^{ev}(G)/(H^{ev}(G) \cap \mathfrak{Nil})$ . As the nilradical  $\mathfrak{Nil}$  is contained in the kernel of every  $k$ -algebra homomorphism from  $H^*(G)$  to  $\Omega$ , the variety  $\text{Hom}_k(H^*(G), \Omega)$  is canonically homeomorphic to the variety  $\text{Hom}_k(H^{ev}(G), \Omega)$ .

In particular, the Quillen Stratification Theorem as stated in [18, Theorem 10.2] and [3, Theorem 5.6.3] can be proved accordingly with our definitions:

**Theorem 3.3** (Quillen's Stratification Theorem). *Let  $\mathcal{A}$  be a set of representatives of the  $G$ -conjugacy classes of elementary abelian subgroups of  $G$ . Then  $V_G$  is the disjoint union*

$$V_G = \coprod_{A \in \mathcal{A}} V_{G,A}^+.$$

*of locally closed subvarieties  $V_{G,A}^+$ . Moreover, for every  $A \in \mathcal{A}$ , the automorphism group  $\text{Aut}_G(A)$  acts freely on  $V_A^+$  and the map  $\text{res}_{G,A}^*$  induces a homeomorphism  $V_A^+/\text{Aut}_G(A) \rightarrow V_{G,A}^+$ .*

The fact that  $V_G = \coprod_{A \in \mathcal{A}} V_{A,G}^+$  for any set  $\mathcal{A}$  of representatives of the  $G$ -conjugacy classes of the elementary abelian subgroups of  $G$ , will be used in our proof in the following form:

**Remark 3.4.** Let  $A$  and  $A'$  be elementary abelian subgroups of  $G$ . If  $A$  and  $A'$  are  $G$ -conjugate then we have  $V_{G,A}^+ = V_{G,A'}^+$ , and if  $A$  and  $A'$  are not  $G$ -conjugate then  $V_{G,A}^+$  and  $V_{G,A'}^+$  are disjoint.  $\square$

*Proof of Theorem C.* Assume that the restriction of the map  $\text{res}_{G,H}^*: V_H \rightarrow V_G$  to the subvariety  $\text{res}_{H,S \cap O^p(G)}^* V_{S \cap O^p(G)}$  of  $V_H$  is injective.

*Step 1:* Let  $A$  be an elementary abelian subgroup of  $S \cap O^p(G)$ . We show that the map  $\text{res}_{G,H}^*$  induces a bijection from  $V_{H,A}^+$  to  $V_{G,A}^+$ . Moreover, if  $A'$  is another elementary abelian subgroup of  $S \cap O^p(G)$  such that  $V_{G,A}^+ = V_{G,A'}^+$ , then we show  $V_{H,A}^+ = V_{H,A'}^+$ .

To see this note that, by Remark 3.1, we have that  $\text{res}_{G,H}^* V_{H,A}^+ = V_{G,A}^+$  and that  $V_{H,A}^+ = \text{res}_{H,S \cap O^p(G)}^* V_{S \cap O^p(G),A}^+$  is contained in  $\text{res}_{H,S \cap O^p(G)}^* V_{S \cap O^p(G)}$ . By a symmetric argument, it follows that  $V_{H,A'}^+$  is contained in  $\text{res}_{H,S \cap O^p(G)}^* V_{S \cap O^p(G)}$  and  $\text{res}_{G,H}^* V_{H,A}^+ = V_{G,A}^+ = V_{G,A'}^+ = \text{res}_{G,H}^* V_{H,A'}^+$ . As we assume that the restriction of  $\text{res}_{G,H}^*$  to  $\text{res}_{H,S \cap O^p(G)}^* V_{S \cap O^p(G)}$  is injective, the above assertion follows.

*Step 2:* Let  $A$  and  $A'$  be two  $G$ -conjugate elementary abelian subgroups of  $S \cap O^p(G)$ . We show that  $A$  and  $A'$  are  $H$ -conjugate. For the proof note that  $V_{G,A}^+ = V_{G,A'}^+$  by Lemma 3.4. So by Step 1, we have  $V_{H,A}^+ = V_{H,A'}^+$ . Thus, again by Lemma 3.4 now used with  $H$  in place of  $G$ , the subgroups  $A$  and  $A'$  need to be  $H$ -conjugate. This completes the proof of Step 2.

*Step 3:* Let  $A$  be an elementary abelian subgroup of  $S \cap O^p(G)$ . We show that  $\text{Aut}_G(A) = \text{Aut}_H(A)$ . By the Quillen stratification theorem Theorem 3.3, the group  $\text{Aut}_G(A)$  acts freely on  $V_A^+$ , and the map  $\text{res}_{G,A}^*$  induces a homeomorphism  $V_A^+ / \text{Aut}_G(A) \rightarrow V_{G,A}^+$ . In particular, the fibres of the map  $\text{res}_{G,A}^*: V_A^+ \rightarrow V_{G,A}^+$  are precisely the orbits of  $\text{Aut}_G(A)$  on  $V_A^+$ . Similarly, applying the Quillen stratification theorem with  $H$  in place of  $G$ , we get that  $\text{Aut}_H(A)$  acts freely on  $V_A^+$ , and the fibres of the map  $\text{res}_{H,A}^*: V_A^+ \rightarrow V_{H,A}^+$  are precisely the orbits of  $\text{Aut}_H(A)$  on  $V_A^+$ . Note that  $\text{res}_{G,A}^* = \text{res}_{G,H}^* \circ \text{res}_{H,A}^*$ . As the map  $\text{res}_{G,H}^*: V_{H,A}^+ \rightarrow V_{G,A}^+$  is by Step 1 a bijection, it follows that the maps  $\text{res}_{G,A}^*: V_A^+ \rightarrow V_{G,A}^+$  and  $\text{res}_{H,A}^*: V_A^+ \rightarrow V_{H,A}^+$  have the same fibres. So the  $\text{Aut}_G(A)$ -orbits on  $V_A^+$  are the same as the  $\text{Aut}_H(A)$ -orbits. As the actions of  $\text{Aut}_G(A)$  and  $\text{Aut}_H(A)$  on  $V_A^+$  are free, this implies that  $|\text{Aut}_G(A)| = |\text{Aut}_H(A)|$ . Thus, since  $\text{Aut}_H(A) \subseteq \text{Aut}_G(A)$ , it follows  $\text{Aut}_G(A) = \text{Aut}_H(A)$ .

*Step 4:* We are now in a position to complete the proof. Let  $A$  and  $A'$  be elementary abelian subgroups of  $S \cap O^p(G)$ . We want to show that  $\text{Hom}_G(A, A') = \text{Hom}_H(A, A')$  and can assume without loss of generality that  $A$  and  $A'$  are  $G$ -conjugate. Then  $A$  and  $A'$  are  $H$ -conjugate by Step 1 and thus there exists  $\psi \in \text{Hom}_H(A, A')$ . Let  $\varphi \in \text{Hom}_G(A, A')$ . Note that  $\varphi = \psi \circ (\psi^{-1} \circ \varphi)$  and  $\psi^{-1} \circ \varphi \in \text{Aut}_G(A) = \text{Aut}_H(A)$  by Step 2. So it follows that  $\varphi \in \text{Hom}_H(A, A')$  which proves  $\text{Hom}_G(A, A') = \text{Hom}_H(A, A')$ . By Puig's hyperfocal subgroup theorem [16, §1.1], we have  $S \cap O^p(G) = \text{hyperfocal}(\mathcal{F}_S(G))$ . So using Theorem A, we can conclude that  $\mathcal{F}_S(G) = \mathcal{F}_S(H)$ . Thus, by the Cartan–Eilenberg stable elements formula [9, XII.10.1], the map  $\text{res}_{G,H}: H^*(G) \rightarrow H^*(H)$  is an isomorphism.  $\square$

**Remark 3.5.** A version of Theorem C can also be formulated and proved for abstract saturated fusion systems rather than for groups. Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ . Assume that  $k$  is an algebraically closed field of characteristic  $p$ . The

cohomology ring  $H^*(\mathcal{F}) = H^*(\mathcal{F}, k)$  of the saturated fusion system  $\mathcal{F}$  is the subring of  $\mathcal{F}$ -stable element in  $H^*(S) = H^*(S, k)$ , which is the subring of  $H^*(S)$  consisting of elements  $\xi \in H^*(S)$  such that  $\text{res}_P^S(\xi) = \text{res}_\varphi(\xi)$  for any  $\varphi \in \text{Hom}_\mathcal{F}(P, S)$  and any subgroup  $P \leq S$ . The ring  $H^*(\mathcal{F})$  is a graded commutative ring. We write  $V_\mathcal{F}$  for the maximal ideal spectrum of  $H^*(\mathcal{F})$ , or alternatively for the variety of  $k$ -algebra homomorphisms from  $H^*(\mathcal{F})$  to  $k$ .

Let  $\mathcal{G}$  be a saturated fusion subsystem of  $\mathcal{F}$ . Note that any  $\mathcal{F}$ -stable element of  $H^*(S)$  is in particular  $\mathcal{G}$ -stable, so we can consider the inclusion map  $\text{res}_{\mathcal{F}, \mathcal{G}}: H^*(\mathcal{F}) \rightarrow H^*(\mathcal{G})$  which then gives us a map  $\text{res}_{\mathcal{F}, \mathcal{G}}^*: V_\mathcal{G} \rightarrow V_\mathcal{F}$  of varieties. Similarly, if  $Q \leq S$ , we are given a  $k$ -algebra homomorphism  $\text{res}_{\mathcal{F}, Q}: H^*(\mathcal{F}) \rightarrow H^*(Q)$  by composing the inclusion map  $H^*(\mathcal{F}) \hookrightarrow H^*(S)$  with the restriction map  $\text{res}_{S, Q}: H^*(S) \rightarrow H^*(Q)$ . Again, this induces a map of varieties  $\text{res}_{\mathcal{F}, Q}^*: V_Q \rightarrow V_\mathcal{F}$ . In particular, if  $A \leq S$  is elementary abelian, one can define  $V_{\mathcal{F}, A}^+ = \text{res}_{\mathcal{F}, A}^* V_A^+$ . In an unpublished preprint, Markus Linckelmann [15, Theorem 1] proves a version of the Quillen stratification theorem; see also Theorem 1.3 and Remark 1.1 in [19]. Using this, one can similarly prove the following version of Theorem C for fusion systems:

Let  $\mathcal{G} \subseteq \mathcal{F}$  be an inclusion of saturated fusion systems over the same finite  $p$ -group  $S$ , and  $p$  an odd prime. If the restriction of the map  $\text{res}_{\mathcal{F}, \mathcal{G}}^*: V_\mathcal{G} \rightarrow V_\mathcal{F}$  to  $\text{res}_{\mathcal{G}, \text{hyp}(\mathcal{F})}^* V_{\text{hyp}(\mathcal{F})}$  is injective, then  $\mathcal{F} = \mathcal{G}$  and in particular  $H^*(\mathcal{F}) = H^*(\mathcal{G})$ .

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